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The transfer matrices of the self-similar fractal potentials on the Cantor set

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Abstract. On the basis of an exact formalism, a system of functional equations for the tunnelling parameters of self-similar fractal potentials (SSFPs) is obtained. Three different families of solutions are found for these equations, two of them having one parameter and one being free of parameters. Both one-parameter solutions are shown to be described, in the long-wave limit, by a fractal dimension. At the same time, the third solution yields transfer matrices which are analytical in this region, similar to the case of structures with the 'Euclidean geometry'. We have revealed some manifestations of scale invariance in the physical properties of SSFPs. Nevertheless, in the common case these potentials do not possess, strictly speaking, this symmetry. The point is that SSFPs in the common case are specified, in contrast to the Cantor set, by two length scales but not one. A particular case when SSFPs are exactly scale invariant to an electron with well defined energy is found.

Introduction

It is known [1] that a common feature of deterministic fractal structures is self-similarity, that is, their geometric properties display scale invariance. Owing to this, such structures are of twofold interest. On the one hand, there is the necessity to study the physical properties of the fractals themselves, which, as follows from experience, are quite frequently encountered in nature. On the other hand, fractals provide a unique possibility to examine physical and geometric aspects of the scale-invariance phenomenon which is known to be observed not only in fractal physics but also plays an important part in the theory of phase transitions and in quantum field theory [2, 3] (we should also draw the reader's attention to [4, 5] where 'self-similar potentials' are investigated; this term, however, has another meaning there because these potentials do not possess fractal geometry). In the first case, the main object is to find to what extent the geometry of fractals influences their physical properties, and whether there exists scale invariance in 'physical fractals'. It is important here to deal with ideal fractals since any approximation made in solving the problem spoils its initial symmetry. As regards the second case, one needs here to solve a problem which is, in some sense, inverse with respect to the first one. That is, starting from the hypothesis of scale invariance of the physical properties, one needs to reconstruct the structure of the system investigated (for example, to construct the Hinsburg–Landau Hamiltonian in order to describe the systems near the phase-transition point [2, 3]). From our viewpoint, it is important here to answer the question 'can continuous non-fractal systems display scale-invariant physical properties?' The point is that, in the phase transitions, the scale-invariance hypothesis was assumed originally without any assumptions regarding the fractality of random fields, that is, fluctuations. Nevertheless, there have been

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attempts to describe phase transitions, for example in spin glasses (see [6]), by making use of the fractal approach.

It seems likely that none of spatial symmetries occurring in modern physics have led to so many problems as has happened in the case of self-similarity. Spatial symmetry in physics problems usually facilitates their solution. However, here we have the opposite situation: to take this symmetry into account correctly is a very difficult task. As far as we know, there is no continuous model of media deterministic fractal, which can be posed and solved rigorously. An important exception is provided by the papers of Lapidus and co-workers (see, for example, [7–9]). However, at present extensive use is made of semi-empirical, approximate analytical methods, or solving the initial problem is reduced to a numerical modelling of the so-called pre-fractals ('real fractals') possessing, in contrast with the fractals, the smallest structure block.

Of special interest is the renormalization group approach. It was originally developed in phase-transition and quantum field theories [2]. This technique is, up to now, the only means of providing reasonable data for such complicated systems. However, it is intuitive in many respects. In particular, the question of justifying the 'scaling hypothesis' remains to be solved. In the last few years a technique with the same name has come into widespread use to describe the so-called self-similar lattices (see, for example, [10]). The lattice variants of the renormalization group method are far simpler than the continuous ones. Therefore, lattice approaches are more rigorous mathematically.

The mathematical problems which appear when one attempts to correctly describe distributed parameter systems involving fractals (continuous models) are most easily understood in the case of one-dimensional structures. In particular, they may be seen in studying wave transmission [11–14] and diffraction (see, for example, [15–17]) in media involving fractals with a Cantor-like geometry. In this paper, our attention is focused on the first example because the tunnelling problem and that of light transmission (the opto-mechanical analogy) [11–13] may be posed correctly even in the one-dimensional case. Also, the simplest diffraction problem [15–17] for a wave incident normally onto a Cantor lattice may be stated rigorously for a plane only.

It should be noted that the physical properties of fractals depend not only on their geometry but also on some functions which characterize them as physical objects. In the tunnelling problem, such a function is a potential. On specifying the function, we thereby introduce into the system investigated a new length scale. As a result, 'a physical fractal', that is a potential with its support (the Cantor set), loses, in the general case, the scale-invariant symmetry which is inherent to geometric fractals, that is, to the Cantor set itself. Since we are interested in studying the scale invariance we will deal with potentials which either fully preserve the symmetry of the fractal geometry, or break it slightly. Among the fractal potentials known in the literature, only the so-called self-similar potential (SSFP) considered in [11–13] may fulfil this role.

As is known (see, for example [12, 13]) an SSFP is constructed together with the Cantor set by means of an infinite iterative procedure where the potential, in an initial step, is approximated by a rectangular barrier (or well). We will consider the case when the interval of the barrier is then divided into three parts. In the extreme (barrier) intervals to be taken here α ($\alpha > 2$) times shorter than the initial one, the potential is then enlarged by a factor of $\alpha/2$, but, in the middle interval, it is equated to zero. This procedure is then repeated indefinitely for all barrier intervals. After n steps have been made, a structure to be referred to further as a pre-fractal of the nth generation [12] is obtained. Such a structure represents the system of rectangular barriers (or wells) whose total area is the same for all generations. The SSFP results from the infinite number of iterations. This potential is singular because the Cantor set where the SSFP differs from zero, has a measure of zero. And its power, that is the integral of the SSFP over the interval where the Cantor set is defined, equals the area of the original rectangular barrier (or well). In contrast to pre-fractals, the SSFP is an ideal structure and is devoid of the smallest structure block. We note that it represents, in turn, the infinite hierarchy of SSFPs of different levels: the original SSFP, that is a zero-level SSFP, consists of two first-level SSFPs; each of them, in turn, have two second-level SSFPs and so on.

As was shown in [11, 12], the true structure of the SSFP is described by recurrence relations for the transfer matrices of SSFPs of different levels. To solve them in the short-wave domain, a variant of the Born approximation was developed (a similar approach was used to solve the diffraction problem [16, 17]; in addition, in [12], the SSFP was presented as a sum of δ-potentials). On this basis an analytical expression for the transmission coefficient was obtained. In the general case the recurrence relations remain to be solved and have therefore been used for numerical modelling of pre-fractals.

In our opinion, at this point we arrive at the most intricate question which occurs in studying the fractal structures: 'can an ideal fractal potential be approximated by structures with the Euclidean geometry?' The basis of the numerical modelling of pre-fractals is usually an implicit assumption that the greater the number of a pre-fractal generation, the more exactly their physical characteristics approximate those of the corresponding ideal fractal. However, as follows from the numerical modelling of electron tunnelling through pre-fractals (see, for example, [13]), the wavenumber dependence of the tunnelling parameters become more irregular as the number of the pre-fractal generation increases. It seems plausible that a limiting structure should be described by non-differentiable functions of the wavenumber. In such cases, the most interesting knowledge which may be derived from the numerical calculations may be information about envelopes (if they exist) of such functions, or their averaged characteristics [13]. In any case, in order to appreciate the difference between the physical properties of the limiting 'pre-fractal' and those of the corresponding ideal fractal, it is very important to develop the theory of structures with the fractal geometry. At present there are strong grounds for believing that this difference should be essential. Let us recall, for example, the paper [13] concerning the approximation of SSFPs by differentiable functions. The authors of [13] concluded that such an approximation may be justified only for sufficiently large values of the wavenumber k. In addition, there is a more radical point of view (see, for example, [18–20] and references therein) in which any function defined on a fractal set should be described by derivatives of a fractional rather than integer order. However, should this be the case then the basic physics equations expressed in differential form should be revised in order to describe the fractal structures, so that in this approach the approximation of a fractal by pre-fractals is most likely to be impossible.

We agree in part with the latter. As will be seen from the following, the physical properties of ideal fractals and pre-fractals of all generations differ qualitatively, since initial symmetry of the problem should be broken when one passes from the ideal structures to the pre-fractals. Thus, the possibility to approximating the SSFP by pre-fractals is, in our view, doubtful. At the same time, despite the attractiveness of the idea based on the fractional derivatives, the description of SSFPs must be developed within the framework of the one-dimensional Schrödinger equation (OSE), because the SSFP structure is taken into account correctly in the recurrence relations [11, 12] for the corresponding transfer matrices (TM). Our aim is to obtain these correlations in a form more suitable for us (this task is solved easily with the help of a technique from [21]), and to solve them.

1. Models of the SSFP: the transfer matrix method

The first difficulty to occur in studying SSFPs lies in the fact that we know neither an explicit expression for such potentials nor how wide is the class of self-similar potentials (the iterative procedure of constructing the SSFP is of secondary importance and cannot serve as its definition). Nevertheless, it is known (see, for example, [12, 17]) that the SSFP can be presented as an eigenfunction of some symmetry operator which is a combination of scale transformations and translations. As will be seen from the following, this knowledge is quite sufficient to obtain the requisite data about the SSFP.

From our viewpoint, a more proper mathematical form of a symmetry condition for the SSFP in the interval $[0, L]$ is given by a functional equation

$$V_0(x) = \tfrac{1}{2}\alpha\,[V_0(\alpha x) + V_0(\alpha(x - x_0))] \tag{1}$$

where $x_0 = L - L/\alpha$. Of course, solving this equation is of particular interest. However, this is beyond the scope of our paper. The point is that all the unknown details about SSFP are concerned only with its behaviour on the Cantor set itself. (In particular, from the iterative procedure widely used for constructing the SSFP, it follows that the potential differs from zero everywhere on the Cantor set. At the same time the case when the self-similar potential is non-zero only on some countable subset of the Cantor set (for example, on its endpoints) must not be ruled out.) However, as will be seen from the following, these details are of no particular importance in finding the tunnelling parameters of SSFPs. One can extract all the required information about the potential defined by equation (1) taking into account only the geometry of regions where it is equal to zero. Therefore, our next step is to find restrictions on the TM sought, which appear due to the symmetry of these domains.

As was shown in [11, 12], the transfer matrix method (TMM) is a proper tool to study potentials defined on the Cantor set. Here we will use the original variant [21] of this method, which is of greater advantage in comparison with the other well known modifications (the recurrence relations, in this method, are obtained directly for observable quantities: for the (real) transmission coefficient and two phases).

So, we suppose that the tunnelling of an electron (a de Broglie wave) through one-dimensional structures with the fractal symmetry is described, as well as any 'Euclidean potential', by the OSE

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \left(k^2 - \frac{2m}{\hbar^2}V_0(x)\right)\Psi = 0 \tag{2}$$

where $k = \sqrt{2mE/\hbar^2}$; E is the electron energy; m is its mass; $V_0(x) \equiv 0$, if $x \notin [0, L]$. It means that, for example, the SSFPs of the first level are positioned in the intervals $[0, L/\alpha]$ and $[L - L/\alpha, L]$, and so on.

It should be noted that in [21] the transfer matrix is a matrix connecting the general solutions of the OSE only in the so-called out-of-barrier regions (OBRs) where a potential is equal to zero. This method is more suitable for solving this problem because in the case of the SSFP, OBRs cover almost the whole OX-axis exclusive of a (Cantor) set of zero measure. Thus, equation (2) may be considered as being solved, if the coefficients of the general solutions in the OBRs were found. In the TMM these coefficients are expressed in terms of the corresponding TMs.

So, for example, let

$$\Psi(x; k) = A_{l,r}^{(+)}(k)\,\mathrm{e}^{\mathrm{i}kx} + A_{l,r}^{(-)}(k)\,\mathrm{e}^{-\mathrm{i}kx} \tag{3}$$

be the general solutions of equation (2) in the intervals $(-\infty, 0)$ (index l) and (L, ∞) (index r).

In line with the TMM [21] these solutions are connected by the relation

$$\mathcal{A}_l = \boldsymbol{Y}_0(k; 0, L)\mathcal{A}_r \qquad \mathcal{A}_{l,r} = \begin{pmatrix} A_{l,r}^{(+)} \\ A_{l,r}^{(-)} \end{pmatrix}$$

where $Y_0(k; 0, L)$ is the TM of the whole fractal, that is, the SSFP of the zero level. The two latter arguments of the matrix point to the fact that it depends not only on the form of the barrier but also on its location on the OX-axis. In the general case the TM $\boldsymbol{Y}(k; x_1, x_2)$ (see [21]) describing the barrier located in the interval $[x_1, x_2]$, may be presented in the form

$$\boldsymbol{Y}(k; x_1, x_2) = \boldsymbol{D}^{-1}(k, x_1)\boldsymbol{Z}(k)\boldsymbol{D}(k, x_2) \tag{4}$$

where

$$\boldsymbol{Z}(k) = \begin{pmatrix} q(k) & p(k) \\ p^*(k) & q^*(k) \end{pmatrix} \qquad \boldsymbol{D}(k, x) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}kx} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}kx} \end{pmatrix}$$
$$q(k) = \frac{1}{\sqrt{T(k)}}\exp(-\mathrm{i}J(k)) \qquad p(k) = \sqrt{\frac{R(k)}{T(k)}}\exp(\mathrm{i}(\tfrac{1}{2}\pi + F(k))) \tag{5}$$

and $R = 1-T$; here $T(k)$, $J(k)$ and $F(k)$ are the tunnelling parameters, that is, the transmission coefficient and phase characteristics of the SSFP. It should be noted that the matrix $\boldsymbol{Z}(k)$ is independent of the barrier location.

We have previously shown [22] that, for a potential barrier which is symmetric relative to the midpoint of the interval where it is positioned, the phase F equals only either zero or π. It is obviously valid for the SSFPs because they possess this symmetry as well. Moreover, as will be shown, the phases of the potential are sufficient to define in the interval $[0, \pi]$. Thus, we only have to search for two of its parameters (the transmission coefficient T and the phase J) supposing that $F = 0$.

2. A hierarchical structure of the SSFP and its properties

Recurrence relations for the transfer matrices

So, the SSFP represents the hierarchy of SSFPs of various levels. Let, for example, V_n be an SSFP of the nth level (these potentials, of rate 2^n, lie in the segments of length d_n, $d_n = L/\alpha^n$). Each SSFP of the nth level represents a symmetrical system of two SSFPs of the $(n+1)$th level. The connection between the nearest levels can be written more easily for the extreme left fragments in the interval $[0, L]$. We have

$$V_n(x) = V_{n+1}(x) + V_{n+1}(x - x_n) \tag{6}$$

where $x_n = x_0/\alpha^n$.

Let $\boldsymbol{Y}_n(k; a, a + d_n)$ be the TM of an SSFP of the nth level, where a is the left-hand boundary of the SSFP (the TMs for all levels also depend on the width L of the zero-level SSFP and on its power W_0). Then, as for any two-barrier structure, the relation

$$\boldsymbol{Y}_n(k; 0, d_n) = \boldsymbol{Y}_{n+1}(k; 0, d_{n+1})\boldsymbol{Y}_{n+1}(k; d_n - d_{n+1}, d_n) \tag{7}$$

must be true (see [21]).

Let us rewrite it, accounting for (4), in the form

$$\boldsymbol{Z}_n(k) = \boldsymbol{Z}_{n+1}(k)\boldsymbol{D}^{-1}(k, d_n - 2d_{n+1})\boldsymbol{Z}_{n+1}(k). \tag{8}$$

In contrast to (7), relation (8) describes all the fractals of a given level wherever they are in the interval $[0, L]$.

The functional equation for the matrix $\boldsymbol{Z}_0$

It should be noted that relation (8) can only be used provided that the TM of some level is known. However, the main difficulty which is encountered in solving (8) is just caused by the fact that there is no smallest structure element in this potential, and, hence, none of these matrices are known. On this point, most research usually comes from the ideal fractal to the pre-fractals with some level being replaced with a rectangular barrier. However, as was pointed out above, such a replacement breaks the link between adjacent levels, which is expressed by relations (8), and, hence, it is unacceptable for us.

Our aim is to state an extra requirement for the TM of some level to be selected, retaining this link. We will consider the first two levels, although we might choose other levels as well. Note that the power of the first-level SSFPs is half that of the zero level. Therefore, it is natural to demand this barrier to be more transparent (to an electron) than the latter. In addition, since both the levels have the same geometry it is also natural to demand the corresponding potential to be described by the same TM.

In order to state the relation sought which satisfies these requirements we will turn to some auxiliary reasoning. Let us consider a singular potential $V(x) = W\delta(x)$ whose tunnelling parameters are defined by

$$T_W = (1 + u^2)^{-1} \qquad J_W = -\tan^{-1}(u)$$
$$F_W = \begin{cases} 0 & \text{if} \quad u \geqslant 0 \\ \pi & \text{if} \quad u < 0 \end{cases} \qquad u = \frac{mW}{\hbar^2 k} \tag{9}$$

(these expressions can be obtained easily, if one takes a proper limit, from the corresponding ones for the rectangular barrier (see [21]); here W is the power of the potential).

As is seen, the δ-potential defines a new length scale l_W in the system: $l_W \sim W^{-1}$. From (9) it follows that the non-trivial tunnelling parameters of the δ-potential obey the following correlations:

$$T_{W/\beta}(k) = T_W(\beta k) \qquad J_{W/\beta}(k) = J_W(\beta k)$$

where β is a positive value. Note that these relations describe the symmetry of the δ-potential. This means that they can be obtained directly from the OSE (or its integral analogue, the Lippmann–Schwinger equation).

For fractals the power W_n of the nth-level SSFP is uniquely related to its width by

$$W_n(d_n) = W_0 \left(\frac{d_n}{L}\right)^s$$

where s is the fractal dimension. So, for the corresponding scale l_n ($l_n \sim W_n^{-1}$) we have $l_n \sim d_n^{1/s}$. We see that the parameters d_n and l_n are non-commensurable. The reason is that the width of a fractal specifies it as an object in the Euclidean metrics (in this case, we deal in essence with the width of the interval on the OX-axis, where the SSFP is positioned). At the same time, its power and the corresponding scale l_n are closely connected with the fractal geometry of its support. Therefore, the de Broglie wavelength of an electron should naturally be compared with $l_n^{1/s}$ rather than with l_n. Finally, to fulfil both the requirements stated above, the sought-for relation may be presented in the following form:

$$\boldsymbol{Z}_1(k) = \boldsymbol{Z}_0\,(\alpha k). \tag{10}$$

Note that for the SSFP, in contrast with the δ-potential, this correlation cannot be obtained, in the common case, from the OSE. It means that this correlation does not describe other levels. That is, in the common case, the SSFP does not possess scale-invariant symmetry.

However, there is a particular case when the SSFP is scale invariant. As is shown in appendix A, it takes place in the limit $\alpha \to 2+0$. In this case, correlation (10) at $\alpha = 2$ may be obtained directly from the OSE, and it is true for all the levels. Thus, the auxiliary condition (10) satisfies, in addition to the two requirements stated above, a third one. Namely, it coincides at $\alpha \to 2+0$ with the symmetry condition, for the limiting SSFP, which follows immediately from the OSE.

So, returning to the common case, we see that, the TMs of both the levels must obey the recurrence relation (8), at $n = 0$, and relation (10). Substituting expression (10) into (8) and using the properties of the matrix $\boldsymbol{D}$ we obtain the equation for the matrix $\boldsymbol{Z}_0$:

$$\boldsymbol{Z}_0(k) = \boldsymbol{Z}_0(\alpha k)\boldsymbol{D}^{-1}(k, \gamma L)\boldsymbol{Z}_0(\alpha k) \tag{11}$$

where $\gamma = 1 - 2/\alpha$. Thus, the additional relation (10) has led us to a closed equation for the TM of the initial SSFP without breaking the link (see (8) at $n = 0$) between the zero and first levels.

It is appropriate at this point to note that (11) relates only to the zero level. The corresponding equations for other levels may be obtained in a similar way, but the argument of matrix $\boldsymbol{D}$ will be different (note that in the limit $\alpha \to 2+0$, $\boldsymbol{D}$ is the unit matrix). It means that in the common case the recurrence relations should be reduced to functional equations which are different for each level. Thus, strictly speaking, there is no scale invariance in physical properties of the SSFPs in the common case. This is finally explained by the fact that each SSFP of the nth level, in contrast to the corresponding Cantor set, is characterized by two scales rather than one. Namely, one of them is the length l_0 related to the power of the SSFP. Another one is the OBR width which is equal to γd_0 (it is precisely this quantity, rather than the fractal width, that enters into equation (11) explicitly). As was pointed out above, both the scales behave differently at the scale transformations.

In the limit $\alpha \to 2+0$, OBRs disappear ($\gamma \to +0$). The nth level of the limiting SSFP is specified only by one length scale that equals l_n, thereby accounting for the scale invariance of the potential. The most interesting fact is that the tunnelling parameters of the limiting SSFP coincide exactly with those of the δ-potential! That is, functions T_W and y_W ($y_W = \frac{1}{2}\pi - J_W$) (9) are solutions to functional equation (11) at $\alpha = 2$.

Direct and inverse recurrence relations for the tunnelling parameters

Once the TM $\boldsymbol{Z}_0$ have been calculated the problem of finding the TMs for other levels arises (except for the case of the limiting SSFP). In order to solve it, we will first rewrite the recurrence relation (8) in terms of the (real) tunnelling parameters. For this purpose it is sufficient to take the recurrence relations obtained in [21] and to apply them to a system of two identical barriers. Considering that $F = 0$, for the SSFP of the nth level we have

$$T_n(k) = \left(1 + 4\frac{R_{n+1}(k)}{T_{n+1}^2(k)}\cos^2(J_{n+1}(k) + \gamma k d_n)\right)^{-1} \tag{12}$$

$$J_n(k) = J_{n+1}(k) + \tan^{-1}\left(\frac{1 - R_{n+1}(k)}{1 + R_{n+1}(k)}\tan(J_{n+1}(k)) + \gamma k d_n\right). \tag{13}$$

(In the following it is convenient to use a new variable y instead of J: $y = \pi/2 - J$.)

Thus, to calculate some TM for a wider fractal, one can use relations (12) and (13). A more complicated situation arises when it is necessary to find the tunnelling parameters for the $(n+1)$th level providing that those of the nth level are known. It can be done only with the help of recurrence relations which are reciprocal to (12) and (13). In the general case it is impossible to find the tunnelling parameters of single barriers of some barrier system uniquely if those of the system itself are known. However, it can be done for systems of two identical barriers. As is shown in appendix B, the sought-for recurrence relations are given by

$$T_{n+1}(k) = \mathcal{L}_1\,(k, T_n(k), y_n(k)) \tag{14}$$

$$y_{n+1}(k) = \mathcal{L}_2\,(k, T_{n+1}(k), y_n(k)) \tag{15}$$

where

$$\mathcal{L}_1\,(k, T_n(k), y_n(k)) = 2\sqrt{T_n(k)}\frac{\sqrt{T_n(k)} + \sin(B(k, y_n(k)))}{1 + T_n(k) + 2\sqrt{T_n(k)}\sin(B(k, y_n(k)))}$$

$$\mathcal{L}_2\,(k, T_{n+1}(k), y_n(k)) = \gamma k d_n$$
$$+\eta\left(\tfrac{1}{2}B(k, y_n(k)) + \sin^{-1}\sqrt{\tfrac{1}{2}T_{n+1}(k) + R_{n+1}(k)\sin^2\left(\tfrac{1}{2}B(k, y_n(k))\right)}\right)$$

$\eta = 1$, if $\sin[2(y_{n+1}(k) - \gamma k d_n)] \geqslant 0$, otherwise $\eta = -1$; the phase $B(k, y_n(k))$ lies in the first quadrant on the complex plane: $\sin(B(k, y_n(k))) = |\sin(y_n(k) - \gamma k d_n)|$. As follows from these relations, the variation of y_n for any n may be treated only in the interval $[0, \pi]$ because changing y_n by π does not spoil the values of T_{n+1} and y_{n+1} in the case.

By making use of recurrence relations (14) and (15) we can now calculate the TMs for all more narrow levels (recall that for the SSFPs of the first level one can use relation (10) as well).

3. Three solutions of the functional equation

Taking into account that the matrix $\boldsymbol{Z}_0$ also depends on L one can always introduce a dimensionless variable ϕ ($\phi = kL$), and then equation (11) can be written as

$$\boldsymbol{Z}_0(\phi) = \boldsymbol{Z}_0(\alpha\phi)\boldsymbol{D}^{-1}(\phi, \gamma)\boldsymbol{Z}_0(\alpha\phi). \tag{16}$$

The parameters L and W_0 in the TM $\boldsymbol{Z}_0$ have been omitted.

The corresponding equations for the tunnelling parameters can be presented in the form (see (12) and (13))

$$T^2(\alpha\phi)R(\phi) = 4R(\alpha\phi)T(\phi)\sin^2(y(\alpha\phi) - \gamma\phi) \tag{17}$$

$$T(\alpha\phi) = (2 - T(\alpha\phi))\tan\left(y(\alpha\phi) - y(\phi)\right)\tan(y(\alpha\phi) - \gamma\phi). \tag{18}$$

Among the solutions of equations (17) and (18) there are trivial ones which correspond to totally transparent and totally opaque barriers. Such solutions will not be considered here because they are unfit for the SSFPs.

In accordance with the common theory of functional equations [23], the existence of solutions of equations (17) and (18) depends essentially on their properties near the point $\phi = 0$. We have found that the solutions may be expanded, in this region, in powers of either ϕ or ϕ^s where $s = \ln(\alpha)/\ln(2)$. There are four different functions to satisfy these equations. Two of them depend on the arbitrary parameter β, and are characterized by the fractal dimension s. Up to first order we have

$$T_0(\phi) = \beta^2\phi^{2s} \qquad y_0(\phi) = \beta\phi^s \tag{19}$$

where the parameter β may be positive or negative. Other two asymptotics correspond to an expansion in integer powers of ϕ, and are determined uniquely. Namely,

$$T_0(\phi) = a^2\phi^2 \qquad y_0(\phi) = b\phi \tag{20}$$

where

$$a = \frac{2(\alpha - 1)}{\alpha^2(\alpha + 1)} \qquad b = -\frac{2}{\alpha^2(\alpha + 1)} \tag{21}$$

$$a = \frac{2(\alpha - 2)}{\alpha^2(\alpha + 2)} \qquad b = 2\frac{(\alpha - 2)(\alpha + 1)}{\alpha^2(\alpha + 2)}. \tag{22}$$

To calculate these functions on the whole OX-axis, we will rewrite equations (17) and (18) in the form solved with respect to $T(\alpha\phi)$ and $y(\alpha\phi)$. For this purpose it is sufficient to use inverse relations (14) and (15). As a result, we have

$$T(\alpha\phi) = \mathcal{L}_1(\phi, T(\phi), y(\phi)) \tag{23}$$

$$y(\alpha\phi) = \mathcal{L}_2(\phi, T(\alpha\phi), y(\phi)). \tag{24}$$

Now let us introduce the auxiliary functions $t_n(\phi)$ and $f_n(\phi)$:

$$t_n(\phi) = \mathcal{L}_1\,(\phi/\alpha, t_{n-1}(\phi/\alpha), f_{n-1}(\phi/\alpha)) \tag{25}$$

$$f_n(\phi) = \mathcal{L}_2\,(\phi/\alpha, t_n(\phi), f_{n-1}(\phi/\alpha)) \tag{26}$$

$n = 0, 1, \ldots$. Then, in accordance with [23], the solution of functional equations (23) and (24) can be written as

$$T(\phi) = \lim_{n\to\infty} t_n(\phi) \qquad y(\phi) = \lim_{n\to\infty} f_n(\phi). \tag{27}$$

Making use of the iterative procedure described by (25)–(27) with functions (19) at the initial step, we obtain two solutions corresponding to positive and negative values of the constant β. The functions t_0 and f_0 given by expressions (20) together with (21) yield another solution of the functional equations, which is devoid of parameters. As regards functions given by (20) together with (22), the iterative procedure (25)–(27) has proved to be unstable relative to them. It means that there are no functions among the solutions of equations (17) and (18), which would have such asymptotics near the point $\phi = 0$.

It should be noted that we could take, as the initial functions $t_0(\phi)$ and $f_0(\phi)$, any arbitrary functions vanishing at $\phi = 0$ (see [23]). In any case, iterative procedure (25)–(27) provides only the three solutions.

4. Numerical results

The numerical calculations of the tunnelling parameters carried out at different values of α and β have revealed some peculiarities in their dependences on ϕ. Figures 1–3 show the transmission coefficient which is a more interesting characteristic of the tunnelling process, for $\alpha = 10$ and $|\beta| = 1$. (It should be noted that since ϕ is a dimensionless parameter the results presented here are valid for all the SSFPs whose zero levels differ by width L, and have the same power W_0.) The first thing which strikes the eye immediately is a discontinuous character of the function $T(\phi)$. A simple analysis of equations (17) and (18) shows that the first derivative of the transmission coefficient, and the phase $y(\phi)$ itself are discontinuous at the points $\alpha\phi_0$ where the points ϕ_0 form the set of roots of the equation $\sin(B(\phi_0), y(\phi_0)) = 0$. Consequently, both of the functions must also be discontinuous at the points $\alpha^n\phi_0$; $n \geqslant 2$.

In the last analysis this feature is a manifestation of the hierarchical structure of the SSFP. In the long-wave limit, all the peculiarities and, in particular, the appearance of the roots ϕ_0, are associated with the SSFP of the zero level. The appearance of the roots $\alpha^n\phi_0$ may be connected to the nth level, if the analogy between the structure of recurrence relation (14) and

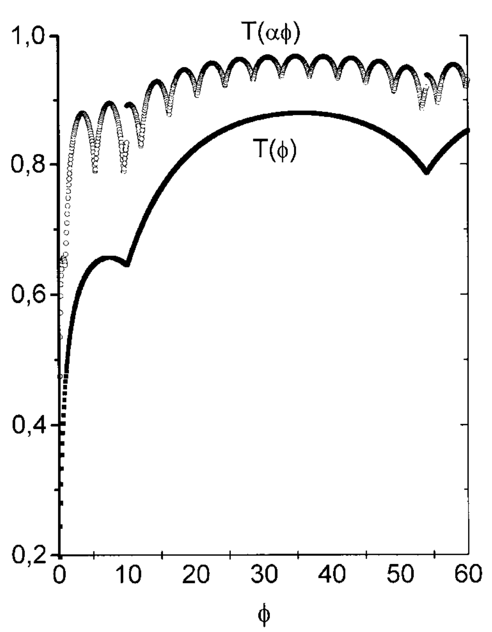

Figure 1. The transmission coefficient T as a function of ϕ for the first solution at $\beta = 1$.

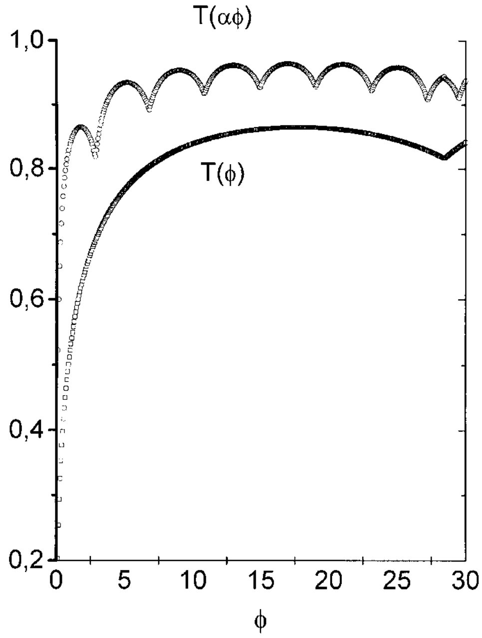

Figure 2. $T(\phi)$ for the second solution at $\beta = -1$.

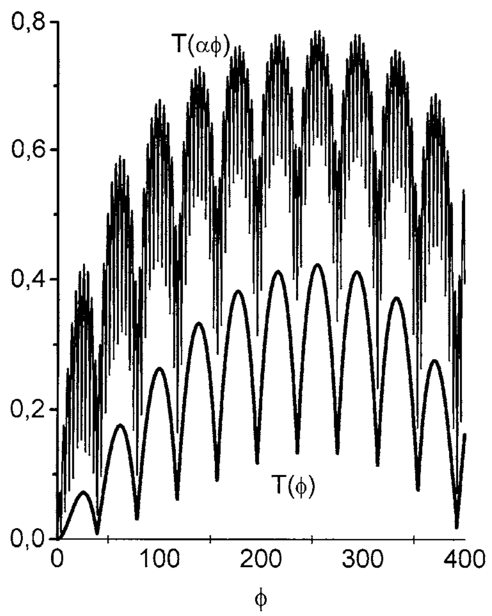

Figure 3. $T(\phi)$ for the third solution.

equation (23) is taken into account (of course, we must remember that, unlike the first level (see (10)), there is no such simple connection between the nth and zero levels). Note that the points ϕ_0 are different for the first two solutions (it is easily checked for the extreme left-hand roots, in the long-wave limit).

A discontinuous character of changing $T(\phi)$ is unusual for structures with Euclidean geometry. Any piecewise-continuous potential bounded by width, as well as the δ-potentials are described by continuous functions $T(k)$. Thus, it seems likely that the problem of electron transport in semi-infinite lattices (see [24]) gives the only example of a 'Euclidean potential' characterized by $T(k)$ having discontinuities (at the boundaries of the forbidden and allowed bands). Although the SSFP is bounded by width, this case is similar, to any extent, to the previous one. So, in the case of the lattice we have an infinite number of periods. For fractals we have an infinite hierarchy of levels. Another interesting analogy lies in the following. We have shown that the behaviour of the tunnelling parameters, for the first two solutions of equations (17) and (18), are characterized by the fractal dimension s. At the same time, for any 'Euclidean potential', exclusive, again, of the semi-infinite lattice, $T(k) \sim k^2$, $y(k) \sim k$ in the long-wave limit. For the lattices we have the following (see [24]): if the point $k = 0$ belongs to a forbidden band, then $T(k) \equiv 0$ in the vicinity of the point; if this point lies in an allowed band, then $T(k) \neq 0$ in the neighbourhood of this point, and the derivatives of $T(k)$ are unbounded here.

As can be seen from the figures, when the width of the interval where $T(\phi)$ is calculated increases by factor of α, so does the number of peaks of this function. This feature can be viewed as a manifestation of the scale invariance. It is particularly clear for the solution with the usual asymptotics in the long-wave region (see figure 3). However, for the solution with the fractal asymptotics, there are distinct deflections from this rule (figures 1 and 2). In all the cases these deflections take place due to the fact that the parameter ϕ enters explicitly in the functional equations. In the long-wave limit the role of this parameter is inessential. The same is true in the particular case when $\alpha \to 2 + 0$. Hence, the limiting SSFP should be scale invariant. And, as is justified by the numerical calculations, the tunnelling parameters of this SSFP are described by (9).

5. Some remarks and conclusions

In the paper a method for solving the recurrence relations for the TMs of the SSFPs is presented. It is shown that their solution can be reduced to solving the functional equation for the TM of the zero level. Three different solutions of this equation were obtained.

Turning our attention to physical aspects of the problem we will point to the following. Firstly, it is shown that the SSFP should be scale invariant in the particular case $\alpha = 2 + 0$ only, its tunnelling parameters coinciding with those of the δ-potentials (it is important to stress here that the fractal dimension of the corresponding Cantor set is equal to unity). In the common case, the SSFP do not, strictly speaking, possess this symmetry: there is no scaling transform to connect the tunnelling parameters of all the adjacent levels of the potential. In the last analysis it is explained by the fact that the fractals are characterized by two length scales rather than one. Nevertheless, in the short- and long-wave regions where one of the scales dominates the scale invariance manifests itself approximately. So, for the first two solutions, in the long-wave limit where the electron wavelength is unlimited together with the sequence of scales $\{l_n\}$, it reveals itself in the power dependence of the tunnelling parameters on ϕ, with the fractal dimension being a scaling exponent. In the short-wave limit, it manifests itself in the fact that the function $\tilde{R}(\phi)$ (the envelope of $R(\phi)$) satisfies the approximate equation $\tilde{R}(\alpha\phi) \approx \tilde{R}(\phi)/4$ (it follows from equation (23) (or (17)) at $R \ll 1$). Thus, the envelope of the reflection coefficient decreases, in the short-wave region, as ϕ^{-2s}.

Secondly, the availability of the three different solutions for the TMs points to the fact that there exist three various kinds of SSFPs. The first two solutions characterized by the fractal dimension may be associated with SSFPs with the positive (barriers) and negative (wells) power W_0. For both the solutions, the role of OBRs in the vicinity of the point $\phi = 0$ is not essential; here $y(\phi) \gg \phi$. The third solution corresponds, in our view, to the SSFP which represents the sum of the δ-potentials arranged at the Cantor set endpoints. In this case, in the long-wave region, we have $y(\phi) \sim \phi$; that is, the phase path of a wave inside the barrier region is of the same order as in the OBRs. This suggests that the whole Cantor set except for the countable subset (where δ-potentials are arranged) consists of the OBRs. These arguments, of course, are not rigorous. Therefore, the question of refining the potential form relative to the three solutions requires further investigations. In particular, a relationship between the parameter β for the first two solutions and the power of the SSFP should be established.

Thirdly, since the transfer matrix for any piecewise-continuous potential of finite width is known to be analytical near the point $k = 0$, we arrive at the conclusion that, at least, in the case of the fractal solutions the corresponding SSFP cannot be approximated by a pre-fractal. In addition, we should make one remark here concerning the investigations carried out on the basis of model equations with fractional derivatives. One of the requirements imposed on these equations, in these approaches, consists of the fact that they must coincide, when the fractal dimension tends to that of the space where the investigated process occurs, with its corresponding conventional analogue. However, as can be seen from our model, for example in the limit $\alpha \to 2+0$, the SSFP does not transform into a structure with Euclidean geometry. The property of the limiting SSFP differs essentially from that of the rectangular barrier which corresponds to the strict equality $\alpha = 2$. In addition, we have to stress that all the required information about the fractal has been obtained here on the basis of the conventional OSE, that is, without fractional derivatives.

Another remark should be made about the correctness of making use of the Born approximation sometimes used in the descriptions of wave scattering in media with fractals. From the functional equations which define the tunnelling parameters of SSFPs, it follows that their behaviour over the k-axis depends essentially on its behaviour in the long-wave region. However, it is the very region where the Born approximation is inapplicable. Therefore, from our viewpoint, making use of this approximation to study ideal fractal structures is questionable.

In summary, let us establish a link between our method and the renormalization techniques mentioned above. This link exists undoubtedly because, in spite of distinctions taking place between them, in all the approaches we deal with the self-similar structures. A simple analysis shows that three main stages may be chosen in our formalism. In the first stage we obtained recurrence relations for the tunnelling parameters of the SSFP. These relations represent, in essence, some map G connecting the TMs of adjacent levels in the structure hierarchy of the SSFP: $\mathbf{Z}_{n+1}(k) = G[\mathbf{Z}_n(k); k]$; $n = 0, 1, \ldots$. In the second stage we supposed that there was a matrix function $\mathbf{Z}_0(k)$ ('fixed points') for which the map G is equivalent to the scale transformation of the independent variable k ('scaling hypothesis'), that is, $\mathbf{Z}_1(k) = G[\mathbf{Z}_0(k); k] = \mathbf{Z}_0(\alpha k)$. The last equality provides a closed functional equation to $\mathbf{Z}_0(k)$. (It should be stressed that for the limiting SSFP the 'scaling hypothesis' transforms into a symmetry condition. In this case relation $\mathbf{Z}_{n+1}(k) = \mathbf{Z}_n(\alpha k)$, for any integer n, may be obtained, as well as the recurrence relations themselves, directly from the OSE.) The third step is dedicated to solving the functional equation. From the mathematical point of view, this problem consists in finding the fixed points of the map presented by auxiliary recurrence relations (25) and (26).

The same three stages may be found in the lattice models. In particular, at the first stage recurrence relations for the physical quantities sought are defined. Their peculiarity lies in the fact that they link physical characteristics of finite lattices (that is, pre-fractals) of consecutive degenerations (at the same time, in our approach, the corresponding recurrence relations link SSFPs of adjacent levels, being themselves ideal fractals). Then, at the second stage, a supposition is made that the recurrence relations representing some map must have invariants (fixed-point orbits). Searching for them is the third stage of such a method.

It is very important to compare our approach with the Vilson renormalization technique [2]. The same three stages may be chosen there, but the order of their priority is different. The reason is that in the first two approaches an exact Hamiltonian used for obtaining recurrence relations is available, whereas, in the phase transition theory, such a Hamiltonian is unknown. Therefore, the starting point for the study is, in fact, the well known hypothesis on the scale invariance of systems in the close vicinity of a critical point. The systems in this region are supposed to be described by the Hinsburg–Landay Hamiltonian which must remain, by hypothesis, unchanged at the scale transformations. The latter change only the Hamiltonian's parameters. Searching for (functional or recurrence) equations describing the change (renormalization) of these parameters at scaling, and their solution are the body of the Vilson formalism.

Let us look at the scaling hypothesis. It is interesting to note that it is stated for large-scale fluctuations which are supposed to play the main role in phase transitions. There are also renormalization group techniques in the field theory, which treat the small-scale transformations. A similar situation exists in our model. Although, strictly speaking, the SSFPs possess scale invariance only in the exceptional case (for the limiting SSFP), there are cases when the scale invariance manifests itself approximately. We have shown that, for the SSFP of the common form, this happens in the long- and short-wave regions. They are the very regions where the renormalization group technique is usually used.

We hope that our study will be useful for further investigations of fractals and scale invariance.

Appendix A. On the scale invariance of the limiting SSFP

Let us consider the case when $\alpha \to 2 + 0$. In other words, let us consider the sequence of SSFPs whose parameter α approaches arbitrarily close to (but not equal to) two. Despite the gaps between Cantor segments, for all levels tending to zero in this case we must obtain, as a result of the infinite iterative procedure, a Cantor set (that is, of zero measure). It is important to emphasize that the limiting Cantor set is of the same measure although its fractal dimension should be unity). Thus the corresponding SSFP remains singular, that is, it does not transform, in the limit, into a rectangular barrier.

As follows from correlations (1) and (6), SSFPs of adjacent levels, with the common left-hand boundary, are connected by means of the transformation

$$V_{n+1}(x) = \tfrac{1}{2}\alpha V_n(\alpha x)$$

for each number n. Although the connection between these potentials is so simple the corresponding TMs cannot, however, be connected (in k-space) by means of some scale transformation which would be true simultaneously at all levels (it is this very fact which demonstrates that SSFPs do not possess scale-invariance symmetry in the general case). The situation is changed significantly in the limit $\alpha \to 2 + 0$, that is, when we have

$$V_{n+1}(x) = V_n(2x). \tag{A1}$$

Let us show that in this case TMs of all the adjacent levels should be connected by a common scale transformation.

As the SSFP is a singular potential, it is more convenient to analyse its symmetry by means of the integral form of the OSE, that is, on the basis of the Lippmann–Schwinger equation (see, for example, [13]). For the zeroth level it can be written as

$$\Psi_0(x,k) = \mathrm{e}^{\mathrm{i}kx} + \frac{1}{2\mathrm{i}k}\int_{-\infty}^{x} \mathrm{d}y\, V_0(y)\Psi_0(y,k)\,\mathrm{e}^{\mathrm{i}k(x-y)}. \tag{A2}$$

Let

$$\tilde{\Psi}_0(x,k) = \Psi_0(x,k)\,\mathrm{e}^{-\mathrm{i}kx}.$$

Then, instead of (A2), we have

$$\tilde{\Psi}_0(x,k) = 1 + \frac{1}{2\mathrm{i}k}\int_{-\infty}^{x} \mathrm{d}y\, V_0(y)\tilde{\Psi}_0(y,k). \tag{A3}$$

Similarly, with the first level we have

$$\tilde{\Psi}_1(x,k) = 1 + \frac{1}{2\mathrm{i}k}\int_{-\infty}^{x} \mathrm{d}y\, V_1(y)\tilde{\Psi}_1(y,k). \tag{A4}$$

(Since the left-hand boundaries of both of these potentials coincide, $\Psi_0(0,k) = \Psi_1(0,k)$.)

It is evident that if in (A4) one makes the change of variables

$$x' = 2x \qquad k' = 2k$$

then, accounting for $V_1(x) = V_0(2x)$, it transforms into (A3). As a result, we have

$$\tilde{\Psi}_1(x,k) = \tilde{\Psi}_0(2x,2k). \tag{A5}$$

We can now find, using the known formula (see, for example, [13]), the complex transmission coefficient. For the potential at hand the required expressions for the coefficients $\tilde{t}_0(k)$ and $\tilde{t}_1(k)$ read as

$$\tilde{t}_0(k) = 1 + \frac{1}{2\mathrm{i}k}\int_{-\infty}^{L} \mathrm{d}y\, V_0(y)\tilde{\Psi}_0(y,k) \tag{A6}$$

$$\tilde{t}_1(k) = 1 + \frac{1}{2\mathrm{i}k}\int_{-\infty}^{L/2} \mathrm{d}y\, V_1(y)\tilde{\Psi}_1(y,k). \tag{A7}$$

Considering in (A7) the relation connecting both the potentials as well as relation (A5) for the wavefunctions, it is easy now to show that $\tilde{t}_1(k) = \tilde{t}_0(2k)$.

Since relation (A1) is valid for any value of n we have

$$\tilde{t}_{n+1}(k) = \tilde{t}_n(2k). \tag{A8}$$

Note now that the complex transmission coefficient for any nth level is connected to the element q_n of the matrix Z_n (see (5)) by relation $\tilde{t}_n = q_n^{-1}$. In the case of the SSFP the matrix element q_n contains all the information about its tunnelling parameters because the phase F_n, in this case, is equal to zero. Thus, relation (A8) shows that the scale invariance exists for the limiting SSFP ($\alpha = 2 + 0$).

Appendix B

Our aim is to solve the system of functional equations (12) and (13) with respect to the values T_{n+1} and y_{n+1}. For convenience, let us introduce the following designations: $\tilde{T} = T_{n+1}$, $\tilde{y} = y_{n+1}$; $\omega = kd_n$. For T_n and y_n the index n will be dropped.

First, let us write down equation (13) in the form

$$\tilde{R} = \cos(2(\tilde{y} - \gamma\omega)) + \sin(2(\tilde{y} - \gamma\omega))\tan(B) \tag{B1}$$

where $B = y - \gamma\omega$, or, otherwise,

$$\eta^2(1 - \mathrm{i}\tan(B)) - 2\tilde{R}\eta + 1 + \mathrm{i}\tan(B) = 0$$

where $\eta = \exp(2\mathrm{i}(\tilde{y} - \gamma\omega))$. Now let us solve this equation with respect to η, choosing the root which behaves correctly at $\tilde{R} = 0$ (see (B1)). As a result, we have

$$\mathrm{e}^{2\mathrm{i}(\tilde{y}-\gamma\omega)} = \left(\tilde{R}\cos(B) + \mathrm{i}\,\mathrm{sgn}(\sin(B))\sqrt{1 - \tilde{R}^2\cos^2(B)}\right)\mathrm{e}^{\mathrm{i}B}. \tag{B2}$$

Let us now introduce a new variable z:

$$\tilde{R} = \frac{\cos(z)}{\cos(\tilde{B})} \qquad 0 \leqslant \cos(z) \leqslant \cos(\tilde{B}) \tag{B3}$$

the phase $\tilde{B}$ lies in the first quadrant on the complex plane; $\cos(\tilde{B}) = |\cos(B)|$. Then, one can show that (B2) is reduced to

$$2(\tilde{y} - \gamma\omega) = \mathrm{sgn}(\sin(2B))\big(z + \tilde{B}\big). \tag{B4}$$

From (B3) we have

$$z = 2\sin^{-1}\sqrt{\tfrac{1}{2}\tilde{T} + \tilde{R}\sin^2\left(\tfrac{1}{2}\tilde{B}\right)}.$$

By taking into account this expression, equation (B4) may be reduced finally to (15).

Now we will transform equation (12). It is noted that

$$\frac{\tilde{T}^2}{\tilde{R}} = \big(\tilde{R}^{-1/2} - \tilde{R}^{1/2}\big)^2.$$

Then, considering (B3) and (B4), equation (12) can be reduced into

$$(\cos(\tilde{B}) - \cos(z))^2 = 4\nu\sin^2\left(\tfrac{1}{2}(z + \tilde{B})\right)\cos(z)\cos(\tilde{B}) \tag{B5}$$

where $\nu = T/R$, or

$$\sin^2\left(\tfrac{1}{2}(z - \tilde{B})\right) = \nu\cos(z)\cos(\tilde{B}). \tag{B6}$$

It is easy to show that this equation, in turn, can be reduced to

$$(Q^2\cos^2(\tilde{B}) + \sin^2(\tilde{B}))\cos^2(z) - 2Q\cos(\tilde{B})\cos(z) + \cos^2(\tilde{B}) = 0$$

where $Q = 1 + 2\nu$.

Let us solve this equation with respect to $\cos(z)$ with the desired root being chosen so that

$$\tilde{R} \equiv \frac{\cos(z)}{\cos(\tilde{B})} = \left(Q + \sqrt{Q^2 - 1}\,\sin(\tilde{B})\right)^{-1}. \tag{B7}$$

With the root ignored the condition $\tilde{R} \leqslant 1$ does not hold. Then, inserting the expression for Q in (B7) we finally obtain equation (14).

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Corrigendum

The transfer matrices of the self-similar fractal potentials on the Cantor set
N L Chuprikov 2000 *J. Phys. A: Math. Gen.* **33** 4293–4308

There is an error in equation (1). The correct form of (1) reads

$$V_0(x) = V_0(\alpha x) + V_0[\alpha(x - x_0)]. \tag{1}$$

This change has an effect on all statements concerning the scale invariance of the SSFP. Now the first relation in appendix A must be replaced by

$$V_{n+1}(x) = V_n(\alpha x).$$

Relation (10) can be extended onto all levels of the SSFP, for any value of n

$$\mathbf{Z}_{n+1}(k) = \mathbf{Z}_n(\alpha k).$$

In this case, for any level n

$$\mathbf{Z}_n(\phi_n) = \mathbf{Z}_n(\alpha\phi_n)\mathbf{D}^{-1}(\phi_n, \gamma)\mathbf{Z}_n(\alpha\phi_n),$$

where $\phi_n = \phi/\alpha^n$. This functional equation is the same for all levels of the SSFP. Hence it is sufficient to find the transfer matrix $\mathbf{Z}_0(\phi)$ from this equation at $n = 0$ (or, equation (16) in the original paper). Then the transfer matrices for the nth level can be found with the help of the relations $\mathbf{Z}_n(\phi_n) = \mathbf{Z}_0(\phi) = \mathbf{Z}_0(\alpha^n\phi_n)$.

There are also two minor errors:

1) the expression $\sin[2(y_{n+1}(k) - \gamma k d_n)] \geqslant 0$, after relation (15), should be replaced by $\sin[2B(k, y_n(k))] \geqslant 0$;

2) the renewed sentence to precede relation (B2), in appendix B, reads 'Now let us solve this equation with respect to η, choosing the root which behaves correctly at $\widetilde{R} = 1 \ldots$' (rather than $\widetilde{R} = 0$).

